# Sensitivity of a storage-ring muon EDM experiment

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## 1 Michel spectrum

The decay distribution of polarized muons is given by the well-known Michel spectrum [1]:

$$\frac{d^2\Gamma}{d\epsilon\, d\cos\theta} = \Gamma_\mu \epsilon^2 [3 - 2\epsilon + P_\mu \cos\theta(2\epsilon - 1)], \tag{1}$$

with

$$\Gamma_\mu \equiv \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{2}$$

the total muon decay rate, $\epsilon = 2E_e/m_\mu$ the reduced positron energy and $\theta$ the polar positron emission angle with respect to the muon spin. The polarization is given by $P_\mu$. The energy spectrum is obtained by integration over $\cos\theta$:

$$\frac{d\Gamma}{d\epsilon} = 2\Gamma_\mu \epsilon^2 (3 - 2\epsilon). \tag{3}$$

In the following we set $\tau_\mu \equiv 1$ for simplicity.

The decay asymmetry $A$ is defined as

$$A(\epsilon, P_\mu) = \mathrm{Prob}(\cos\theta > 0) - \mathrm{Prob}(\cos\theta < 0). \tag{4}$$

Integration of Eq. 1 over the polar angle gives

$$A(\epsilon, P_\mu) = P_\mu \epsilon^2 (2\epsilon - 1). \tag{5}$$

For the average asymmetry, we obtain

$$\langle A \rangle (P_\mu) = \int_0^1 A(\epsilon, P_\mu) d\epsilon = \frac{P_\mu}{6}. \tag{6}$$

## 2 EDM-induced spin rotation

In a muon EDM experiment based on the frozen-spin technique [2] an up-down asymmetry of the decay positrons is measured as a function of the individual lifetimes of longitudinally polarized muons that traverse the detection volume perpendicularly to the up-down axis. Initially, the measured up-down asymmetry is zero, but a finite EDM $\eta$ leads to a rotation of the muon spin out of the orbital plane according to [2]

$$\omega_e t = \frac{\eta}{2}\frac{e}{m}\beta B t, \tag{7}$$

which will manifest itself as a growing up-down asymmetry $\mathcal{A}(t)$ in the detector:

$$\mathcal{A}(t) = AP_\mu \sin(\omega_e t) \approx AP_\mu \omega_e t, \tag{8}$$

where $A = A(P_\mu = 1)$ is the intrinsic asymmetry of the muon decay discussed in the previous section.

## 3 Measurement averaged in time and energy

With Eqs. 7 and 8 the EDM $\eta$ can be extracted from any observed $\mathcal{A}(t)$ according to

$$\eta = \frac{2}{t(e/m)\beta BAP_\mu}\mathcal{A}(t) \tag{9}$$

with error

$$\sigma_\eta = \frac{2}{t(e/m)\beta BAP_\mu}\sigma_{\mathcal{A}(t)} = \frac{2}{t(e/m)\beta BAP_\mu\sqrt{N}}, \tag{10}$$

where we have used the usual statistical error on a counting asymmetry based on $N$ events in the limit of small asymmetry:

$$\sigma_{\mathcal{A}} = \sqrt{\frac{1-\mathcal{A}^2}{N}} \simeq \frac{1}{\sqrt{N}}. \tag{11}$$

Accumulating the asymmetry over all muon lifetimes will yield an average asymmetry for the average decay time $\langle t\rangle = \gamma\tau$, so that the error on $\eta$ becomes

$$\sigma_\eta = \frac{2}{\gamma\tau(e/m)\beta BAP_\mu\sqrt{N}}. \tag{12}$$

If the energy of the emitted positron is not recorded, then $\langle A\rangle = 1/6$ and

$$\sigma_\eta = \frac{12}{\gamma\tau(e/m)\beta BP_\mu\sqrt{N}}. \tag{13}$$

In the following, we will assume the parameters for the proposed PSI compact storage ring setup: $\beta = 0.764$, $\gamma\tau = 3.403$ $\mu$s, $P_\mu = 0.9$ ($\mu$E1 125 MeV/$c$ muon beam line), and $B$ =1 T.

## 4 Measurement in time bins

Measuring the asymmetry as a function of time improves the sensitivity by a factor $\sqrt{2}$, if the initial time $t_0$ is known and the acceptance is constant in decay time. If both the slope and the $t_0$ have to be determined from the asymmetry data, the statistical error on $\eta$ from many asymmetry measurements at various times is equivalent to the error obtained from a single averaged measurement using the same data.

This can be seen explicitly, for example by considering the error on the slope $b$ obtained from a least-squares fit of a straight line $y = a + bx$ through data points ($x_i$,$y_i \pm \sigma_i$) [3]:

$$\sigma_b^2 \approx \frac{\sum_i \frac{1}{\sigma_i^2}}{\left(\sum_i \frac{1}{\sigma_i^2}\right)\left(\sum_i \frac{x_i^2}{\sigma_i^2}\right) - \left(\sum_i \frac{x_i}{\sigma_i^2}\right)^2} \tag{14}$$

In our case we will have $N_i = \frac{N}{\gamma\tau}\exp(-\frac{t_i}{\gamma\tau})\Delta t$ events in bin $i$ stretching from $t$ to $t + \Delta t$, so that the asymmetry measured in this bin will carry a statistical error of $1/\sqrt{N_i}$. Inserting this into Eq. 14 we get, in the limit of infinitesimally small bins,

$$\sigma_{\rm slope}^2 \approx \frac{\gamma\tau}{N} \times \frac{\int e^{-t/\gamma\tau}dt}{(\int e^{-t/\gamma\tau}dt)(\int t^2 e^{-t/\gamma\tau}dt) - (\int t e^{-t/\gamma\tau}dt)^2} = \frac{1}{(\gamma\tau)^2 N}, \tag{15}$$

or, for the error on the asymmetry averaged over all decay times,

$$\sigma_{\mathcal{A}} = \gamma\tau\sigma_{\rm slope} = \frac{1}{\sqrt{N}}, \tag{16}$$

in agreement with Eq. 11, i.e., no gain in sensitivity.

If, however, the initial time is known, we may set $a \equiv 0$ in our fit function, which then reduces to $y = bx$. In this case, Eq. 14 simplifies to

$$\sigma_b^2 \approx \frac{1}{\left(\sum_i \frac{x_i^2}{\sigma_i^2}\right)^2}, \tag{17}$$

therefore,

$$\sigma_{\rm slope}^2 \approx \frac{\gamma\tau}{N} \times \frac{1}{(\int t^2 e^{-t/\gamma\tau}dt)} = \frac{1}{(2\gamma\tau)^2 N}, \tag{18}$$

and

$$\sigma_{\mathcal{A}} = \gamma\tau\sigma_{\rm slope} = \frac{1}{\sqrt{2N}}. \tag{19}$$

With this, Eq. 12 becomes

$$\sigma_\eta = \frac{\sqrt{2}}{\gamma\tau(e/m)\beta BAP_\mu\sqrt{N}} \tag{20}$$

in accordance with Ref. [2].

## 5 Measurement in energy bins

If the detector is capable of measuring the energy of the decay positron (in the muon center-of-mass frame), then we can exploit the energy-angle correlation of the Michel spectrum. In one energy bin $i$, stretching from $\epsilon$ to $\epsilon + \Delta\epsilon$ we can expect to gather $N\epsilon^2(3 - 2\epsilon)\Delta\epsilon$ events (Eq. 3), if $N$ is the total number of recorded events and the acceptance is independent of $\epsilon$. These events will exhibit the spin-asymmetry given by $A(\epsilon) = \epsilon^2(2\epsilon - 1)$ (Eq. 5) and therefore yield an $\eta$-measurement with error

$$\sigma_i = \frac{\sqrt{2}}{\gamma\tau(e/m)\beta BP_\mu\sqrt{N}\epsilon^3(2\epsilon - 1)\sqrt{(3 - 2\epsilon)\Delta\epsilon}}. \tag{21}$$

The errors from many such measurements add up according to

$$\sigma^2_{\text{tot}} = \frac{1}{\sum_i(1/\sigma_i^2)}, \tag{22}$$

therefore, with

$$\sum_i \epsilon^6(2\epsilon - 1)^2(3 - 2\epsilon)\Delta\epsilon \quad \xrightarrow{\Delta\epsilon\to 0} \quad \int_0^1 \epsilon^6(2\epsilon - 1)^2(3 - 2\epsilon)d\epsilon = \frac{127}{1260} \tag{23}$$

we have

$$\sigma_\eta = \frac{\sqrt{2}}{\gamma\tau(e/m)\beta BP_\mu\sqrt{N}} \times \sqrt{\frac{1260}{127}}. \tag{24}$$

In other words, in the very best case (infinitesimally small energy bins over the entire spectrum), an energy-binned measurement leads to an error on $\eta$ as given by Eq. 20 with

$$A_{\text{eff}} = \langle A\rangle\sqrt{\frac{127}{35}} = \frac{1}{6}\sqrt{\frac{127}{35}} \approx 0.31748\ldots \tag{25}$$

A lower threshold on the positron energy initially has very little effect on $A_{\text{eff}}$, since the intrinsic asymmetry is below 10% up to $\epsilon \approx 0.63$. This is illustrated in Fig. 1. We see that $A_{\text{eff}}$ drops to 0.315, 0.31 and 0.3 (the value assumed in Ref. [2] for $\epsilon$ >0.703, 0.762, and 0.815, respectively.

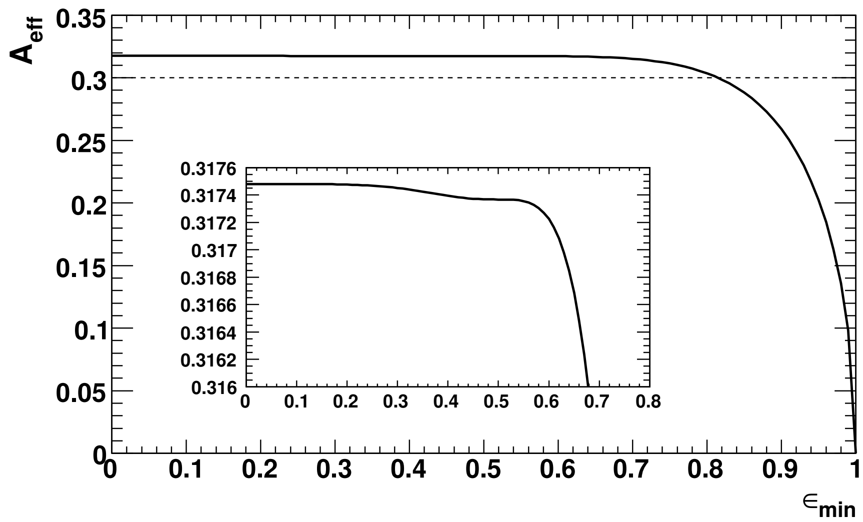


Fig. 1. The effective asymmetry $A_{\text{eff}}$ as a function of the minimum reduced energy $\epsilon_{\text{min}}$. The dashed line marks the assumption of Ref. [2].

## 6 Toy Monte-Carlo study

A toy Monte-Carlo study was performed in order to study the sensitivity of the experiment as a function of parameters less accessible to analytical treatment. The study is based on the fast simulation tool RooFit [4], which is itself based on the ROOT object-oriented data analysis framework [5]. The fitter used by ROOT as well as RooFit is MINUIT [6].

Four variables are used in generating and fitting events:

- $\cos\Theta \in [-1, 1]$: the cosine of the polar angle of the decay positron momentum with respect to the muon orbit direction;
- $\Phi \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$: the azimuthal angle of the decay positron momentum measured relative to the plane tangential to the orbit and perpendicular to the orbital plane, such that positrons reaching the upper (lower) detector have $|\Phi| < (>) \frac{\pi}{2}$;
- $\epsilon \in [0, 1]$: the reduced energy of the positron;
- $t \in [0, \infty]$: the decay time of the muon.

After a spin rotation $\alpha = \omega_e t$, the relation between the angles $\theta$, $\phi$, measured relative to the spin direction, and the angles $\Theta$, $\Phi$, measured relative to the momentum direction is given by

$$\cos\Theta = \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi, \tag{26}$$

$$\tan\Phi = \frac{\sin\theta\sin\phi}{\cos\alpha\sin\theta\cos\phi + \sin\alpha\cos\theta}, \tag{27}$$

with inverse

$$\cos\theta = \sin\alpha\sin\Theta\cos\Phi + \cos\alpha\cos\Theta, \tag{28}$$

$$\tan\phi = \frac{\sin\Theta\sin\Phi}{\cos\alpha\sin\Theta\cos\Phi - \sin\alpha\cos\Theta}. \tag{29}$$

A probability density function (pdf) $W(\cos\Theta, \Phi, \epsilon, t)$ is defined according to Eqs. 1, 26, 27 and an exponential decay law $\propto \exp(t/\gamma\tau)$, with $P_\mu$, $\eta$, $\gamma\tau$ as external parameters. In addition, the pdf keeps a binary (category, tag) variable, which is set to "up" or "down" according to whether $|\Phi| <$ or $> \frac{\pi}{2}$.

In principle the pdf $W$ can be used to random-generate events directly. In practice, however, this is extremely inefficient, as RooFit uses a brute-force accept-reject procedure for custom-made pdfs, which takes a very long time to generate a reasonable number of events in a four-dimensional space. Therefore it is much more efficient to first generate $(\cos\theta, \epsilon)$ according to the Michel spectrum (Eq. 1) along with a flat $\phi$ distribution, then rotate to $(\cos\Theta, \Phi)$ according to Eqs. 26 and 27, and add the separately generated exponential decay time. Figure 2 shows the resulting event distributions for $10^5$ events generated with the parameters given in Sec. 3 and for $\eta = 0$ (top) and with a finite $\eta = 5 \times 10^{-5}$ (bottom) for illustration.

In principle (again...) we should be able to fit our pdf $W$ directly to the generated dataset to determine $\eta$ and its error. Alas, it turns out that such a four-dimensional fit is numerically not stable and fails to converge. Instead, we reduce the dataset to two variables, the decay time $t$ and the up/down tag, to form the time-dependent asymmetry as it would be observed by the experiment. An attempt to fit a corresponding projection of the pdf $W$, i.e., $W$ integrated over all variables except $t$ and tag, to the asymmetry fails again due to numerical instabilities. The many integrations lead to fluctuations that prevent the fit from finding a stable minimum. Increasing the numerical precision of the integration to a level that would be tolerable to the fitter results in excruciating performance penalty. As a work-around we use the projection $W(t, \text{tag})$ only to extract the relation between the slope of the asymmetry and the parameter $\eta$. To reduce numerical noise, we sample the asymmetry produced by $W(t, \text{tag})$ for a given $\eta$ at several points along the time axis and form an average conversion factor. This factor is then used in a simple straight-line fit to the asymmetry to obtain $\eta$ with a statistical error $\sigma_\eta$.

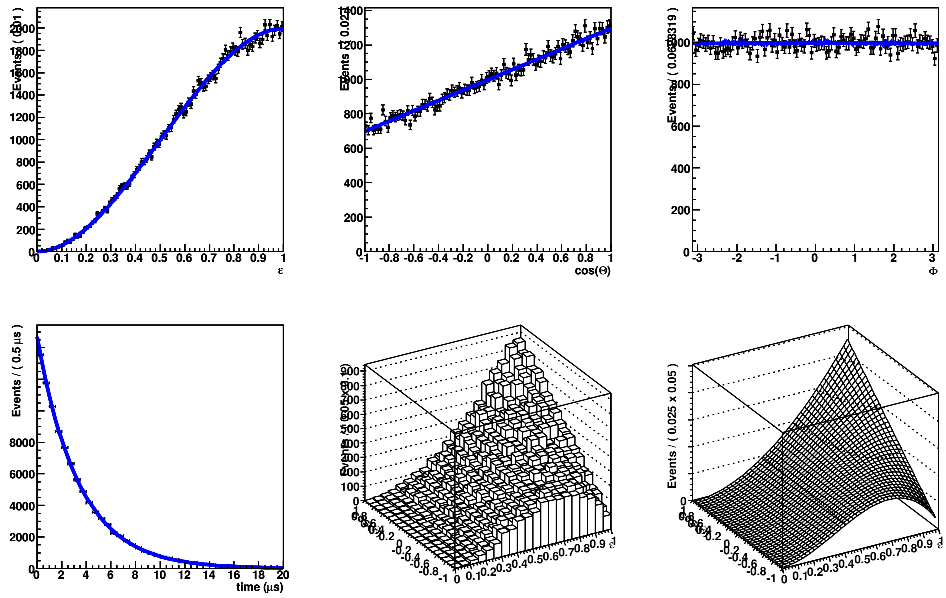


Fig. 2. RooFit output for the distributions in $\epsilon$, $\cos\Theta$, $\Phi$, and $t$ obtained from a fast simulation of $10^5$ events as described in the text, for $\eta = 0$. Data points are event yields per bin, the blue curves represent the pdf. The last two plots show the correlation between $\epsilon$ and $\cos\Theta$, in event yields and pdf, respectively.

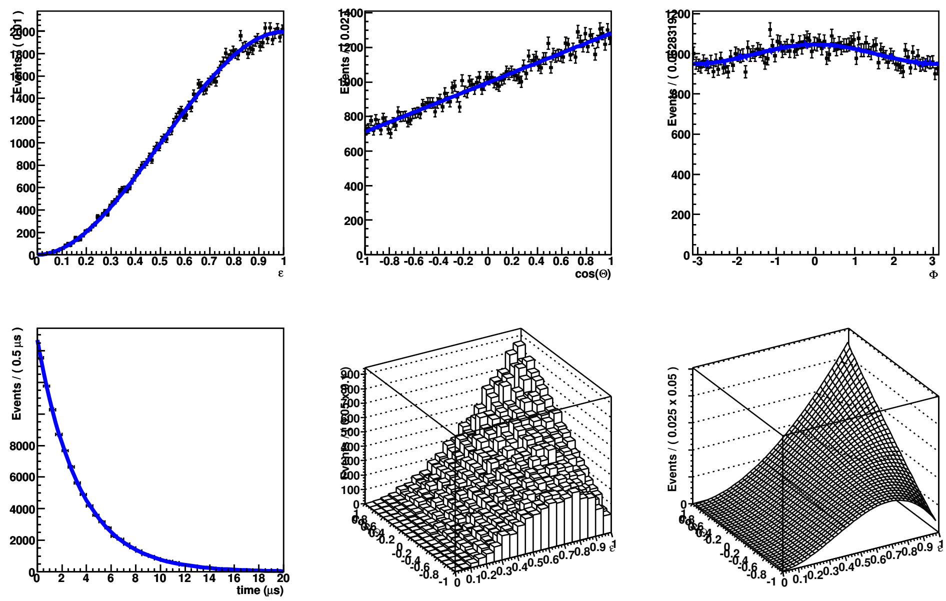


Fig. 3. The same as Fig. 2, but with (huge) $\eta = 2 \times 10^{-4}$. The effect on $\Phi$ is clearly visible.

Figure 4 shows the data asymmetry resulting from a sample of $10^7$ events generated with $\eta = 5 \times 10^{-6}$, corresponding to the current limit [7] with fit function overlaid. Events with $t > 20\ \mu$s are dropped from the fit. For illustration, the RooFit model containing numerical noise is also shown. The plot demonstrates that, from a purely statistical point of view, a muon EDM of the size of the current limit could already be detected with 10 million events, i.e. after 50 seconds of data-taking at a rate of 200 kHz. (An EDM of this size would be required for an EDM-only interpretation of the $g-2$ discrepancy observed by the Brookhaven Muon $g-2$ Collaboration [8].) Note that this result is obtained assuming no information on the positron energy, i.e. effectively with $A = 1/6$.

The error on $\eta$ obtained with $10^7$ events without energy binning is $\sigma_\eta = 1.40 \times 10^{-6}$ to be compared with the result from Eq. 20 with $A = 1/6$, $\sigma_\eta = 1.36 \times 10^{-6}$. Adding a minimum energy requirement $\epsilon > 0.7$ increases the intrinsic decay asymmetry and improves the statistical precision to $\sigma_\eta = 1.08 \times 10^{-6}$ (Fig. 5).

With 50 energy bins ranging from $\epsilon = 0$ to 1, the error shrinks to $\sigma_\eta = 0.98 \times 10^{-6}$. The improvement gained from the energy-sampling turns out to be somewhat smaller than what we would expect from Eq. 25 (error reduction of 0.70 versus $\sqrt{35/127} \approx 0.52$). The reason for this discrepancy is currently under study.

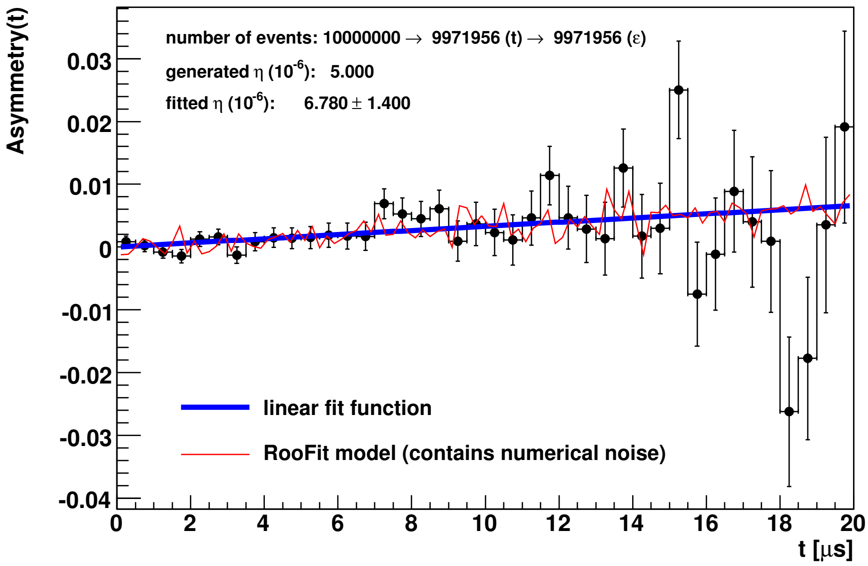


Fig. 4. EDM asymmetry as a function of decay time from $10^7$ generated events for $\eta = 5 \times 10^{-6}$ (corresponding to the current limit [7]). The blue curve is the linear fit function, the thin red curve is the projected pdf, which suffers from numerical noise from the projection integrations but is used to convert the fitted slope to a value for $\eta$. The asymmetry is integrated over all positron energies, corresponding to the case where the detector does not record the energy of the positron (but does record *all* decays). All other parameters according to the PSI setup, see Sec. 3

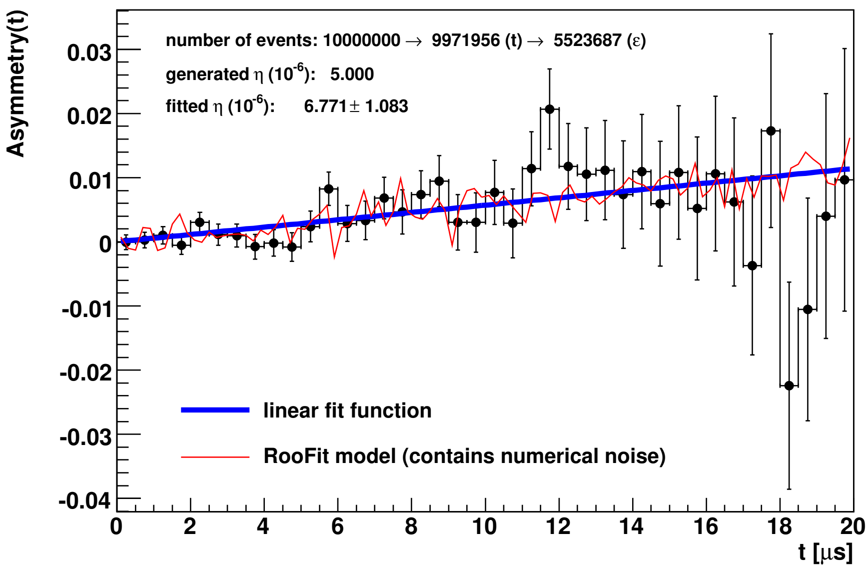


Fig. 5. The same as Fig. 4 but with an energy threshold $\epsilon > 0.7$ (applied in the muon center-of-mass system). The sensitivity improves thanks to the larger intrinsic asymmetry.

## References


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